

MATHEMATICAL MODEL OF COMPARTMENTAL EPIDEMIC

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ABSTRACT

In this paper, we study a model of compartmental epidemic in which the immunity for the recovery class is zero and they move back into the susceptible class. We observed that there are two equilibria points and both equilibria points are under control.

KEYWORDS: Compartmental Epidemic, Equilibria Point, Immunity

1. INTRODUCTION

There are many studies on HIV/AIDS which includes Diekmann et al (2009), Ayeni et al (2010), Oluyo (2013). All these researchers studied different ways of controlling the spread of this epidemic disease. Of a particular interest is Diekmann (2009), who constructed next matrices for compartmental epidemic models on which our own work is based. We revisited his work and considered a case when the recovery class does not have immunity and they were not removed from the system. We observed that the two equilibria points are stable unconditionally and the per capital birth and death rate (μ) has effect on the population but no effect on the stability of the equilibrium point.

2. THE MODEL

We modified Diekmann O et al (2009) model by considering a case when the recovery class does not have immunity and they were not removed from the system.

Arising from the above, the relevant mathematical equations are:

$$\begin{aligned} \frac{dS}{dt} &= \mu N - \frac{\beta SI}{N} - \mu S + \mu R \\ \frac{dE_1}{dt} &= \frac{p\beta SI}{N} - (v_1 + \mu)E_1 + p\mu R \\ \frac{dE_2}{dt} &= (1-p)\frac{\beta SI}{N} - (v_2 + \mu)E_2 + (1-p)\mu R \\ \frac{dI}{dt} &= v_1E_1 + v_2E_2 - (\gamma + \mu)I + (v_1 + v_2)R \\ \frac{dR}{dt} &= \gamma I - \mu R \end{aligned} \tag{1.1}$$

Where S susceptible, E_1 latently infected category 1, E_2 latently infected category 2, I infectious, R recovery/ not removed /no immune, β transmission rate, μ the per capital birth and death rate, p ratio of E_1 in the population, (1-p) ratio of E_2 in the population, v_1 rate of leaving E_1 , v_2 rate of leaving E_2 , γ rate of leaving the infectious state, $N=S+E_1+E_2+I+R$

3. THE CRITICAL POINTS

There exists the following two equilibria correspond to the system (1)

- $E_0(185,0,0,0,0)$
- $E_1(-2143.141172, 208.8163656, 0, -0.2105743539, -110.4971052)$

Stability Criteria

Theorem 1: The equilibria E_0 and E_1 are asymptotically stable unconditionally.

Proof: The general variation matrix M corresponding to system (1.1) is

$$M = \begin{pmatrix} -\mu & 0 & 0 & 0 & \mu \\ 0 & -(v_1 + \mu) & 0 & 0 & p\mu \\ 0 & 0 & -(v_2 + \mu) & 0 & (1-p)\mu \\ 0 & v_1 & v_2 & -(\gamma + \mu) & (v_1 + v_2) \\ 0 & 0 & 0 & \gamma & -\mu \end{pmatrix}$$

At equilibrium point $E_0(185,0,0,0,0)$ the variation matrix M_0 is given by

$$M_0 = \begin{pmatrix} -\mu & 0 & 0 & 0 & \mu \\ 0 & -(v_1 + \mu) & 0 & 0 & p\mu \\ 0 & 0 & -(v_2 + \mu) & 0 & (1-p)\mu \\ 0 & v_1 & v_2 & -(\gamma + \mu) & (v_1 + v_2) \\ 0 & 0 & 0 & \gamma & -\mu \end{pmatrix}$$

And the corresponding eigenvalues to the matrix above are $\lambda_1 = -0.1$, $\lambda_2 = -0.11$, $\lambda_3 = -0.076799353$, $\lambda_4 = -0.1266010032 + 0.018007705521i$, $\lambda_5 = -0.1266010032 - 0.018007705521i$, which shows that the disease free equilibrium point is asymptotically stable.

At equilibrium point $E_1(-2143.141172, 208.8163656, 0, -0.2105743539, -110.4971052)$ the variation matrix M_1 is given by

$$M_1 = \begin{pmatrix} 0.2105743539 \frac{\beta}{N} - \mu & 0 & 0 & 2143.141172 \frac{\beta}{N} & \mu \\ -0.2105743539 \frac{p\beta}{N} & -(v_1 + \mu) & 0 & -2143.141172 \frac{p\beta}{N} & p\mu \\ 0 & 0 & -(v_2 + \mu) & 0 & (1-p)\mu \\ 0 & 0 & v_2 & -(\gamma + \mu) & (v_1 + v_2) \\ 0 & 0 & 0 & \gamma & -\mu \end{pmatrix}$$

And the corresponding eigenvalues to the matrix above is $\lambda_1 = -0.099965280$, $\lambda_2 = -0.08763932023$, $\lambda_3 = -0.11$, $\lambda_4 = -0.11$, $\lambda_5 = -0.1323606798$, which shows that the endemic equilibrium point is asymptotically stable.

Simulations: The system (1.1) is solved numerically. Since infection takes into account the births and death of human.

In the numerical solution the initial value used are $S(0)=100$, $E_1(0)=30$, $E_2(0)=40$, $I(t)=10$, $R(t)=5$ and the

following data were used $\beta=0.03$, $\mu=0.1$, $\gamma=0.02$, $p=1$, $v_1=0.01$, $v_2=0.01$ and the per capital birth and death rate has effect on the population of Susceptible and infectious class but no effect on the stability of the equilibria points.

4. RESULTS

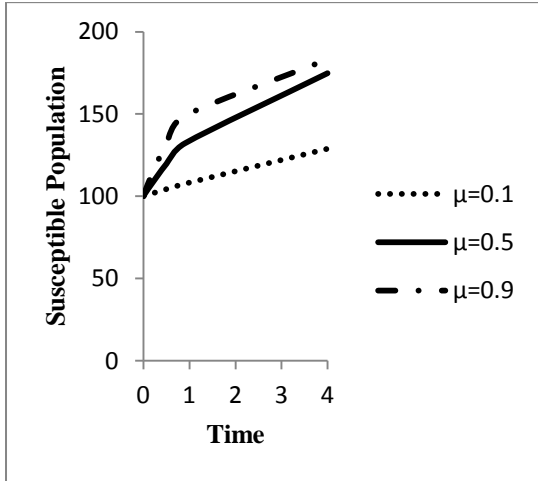


Figure 1: Susceptible Population (W) of E_0 with Time for Various Values of M

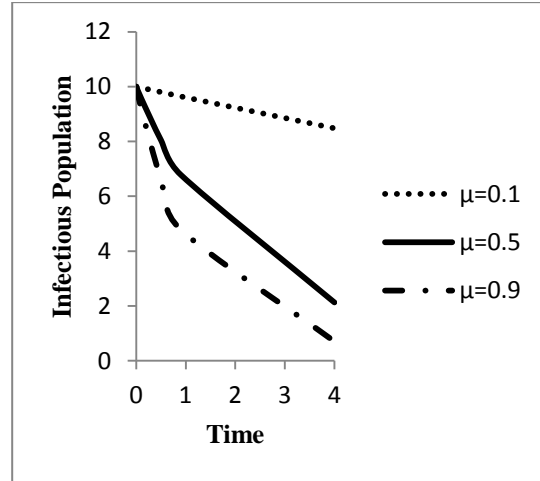


Figure 2: Infectious Population (Z) of E_0 with Time at Various Values of M

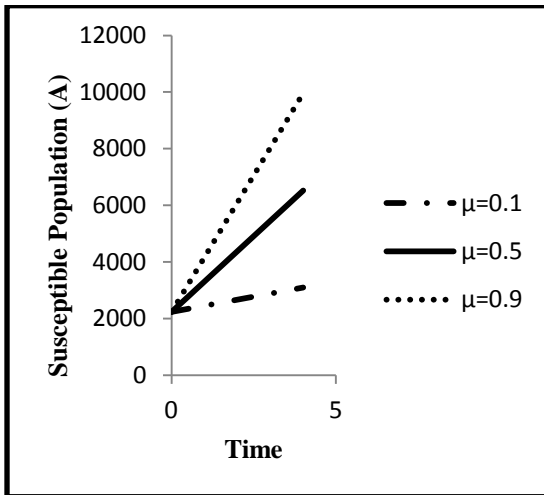


Figure 3: Susceptible Population (A) of E_1 with Time for Various Values of M

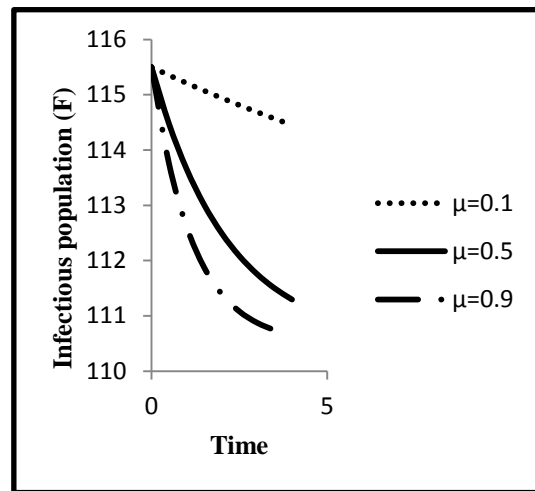


Figure 4: Infectious Population (F) of E_1 with Time at Various Values of M

5. DISCUSSIONS OF RESULTS

From the solution obtained numerically, we observed that increment in per capital birth and death rate (μ) populates the susceptible class and depopulates the infectious class and also we observed that it has no effect on the stability of the two equilibria points.

REFERENCES

1. Diekmann, O, Heesterbeek, J.A.P and Robert M.G (2009): The construction of next- generation matrices for the compartmental epidemic models. Journal of the royal society interface 7, 873-885.
2. Oluyo T.O (2013): Mathematical Analysis of viral and immune system. International journal of Pure and Applied Mathematics Volume 6 No 2. pp 85-92.

3. Diekmann, O. and Heesterbeek, J.A.P (2000): *Mathematical Epidemiology of infectious diseases: model building, analysis and interpretation*. Chichester, UK: Wiley.
4. Diekmann, O, Heesterbeek, J.A.P and Metz, J.A.J (1990): On the definition and computation of basic reproduction number R_0 in models for infectious diseases in heterogeneous populations. *J. Math. Biol.* 28, 365-382.
5. Thieme, H.R (2009): Spectral bound and reproduction number for infinite – dimensional population and time heterogeneity. *SIAM J. Appl. Math.* 70 188-211.